习题　12.2　多元复合函数的求导法则

1．　利用链式规则求偏导数：

（1） $z=\tan(3t+2x^2-y^2),\quad x=\dfrac{1}{t},\quad y=\sqrt{t}$，求$\dfrac{\mathrm{d}z}{\mathrm{d}t}$；

（2） $z=e^{x-2y},\quad x=\sin t,\quad y=t^3$，求$\dfrac{\mathrm{d}^2z}{\mathrm{d}t^2}$；

（3） $w=\dfrac{e^{ax}(y-z)}{a^2+1}$，$y=a\sin x$，$z=\cos x$，求$\dfrac{\mathrm{d}w}{\mathrm{d}x}$；

（4） $z=u^2\ln v,\quad u=\dfrac{x}{y},\quad v=3x-2y$，求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$；

（5） $u=e^{x^2+y^2+z^2}$，$z=y^2\sin x$，求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}$；

（6） $w=(x+y+z)\sin(x^2+y^2+z^2)$，$x=te^s$，$y=e^t$，$z=e^{s+t}$，求$\dfrac{\partial w}{\partial s},\dfrac{\partial w}{\partial t}$；

（7） $z=x^2+y^2+\cos(x+y)$，$x=u+v$，$y=\arcsin v$，求$\dfrac{\partial z}{\partial u},\dfrac{\partial^2 z}{\partial v\partial u}$；

以下假设f具有二阶连续偏导数。

（8） $u=f\left(xy,\dfrac{x}{y}\right)$，求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial^2 u}{\partial x\partial y},\dfrac{\partial^2 u}{\partial y^2}$；

（9） $u=f(x^2+y^2+z^2)$，求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z},\dfrac{\partial^2 u}{\partial x^2},\dfrac{\partial^2 u}{\partial x\partial y}$；

（10） $w=f(x,y,z)$，$x=u+v$，$y=u-v$，$z=uv$，求$\dfrac{\partial w}{\partial u},\dfrac{\partial w}{\partial v},\dfrac{\partial^2 w}{\partial u\partial v}$。

解 (1) 记$u=3t+2x^2-y^2$，则

$$\frac{dz}{dt}=\frac{dz}{du}\frac{du}{dt}=\frac{dz}{du}\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}\right)$$

$$=[3+4x\cdot(-\frac{1}{t^2})-2y\cdot\frac{1}{2\sqrt{t}}]\sec^2 u$$

$$=(2-\frac{4}{t^3})\sec^2(2t+\frac{2}{t^2})。$$

(2) $\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}=e^{x-2y}\cos t-2e^{x-2y}\cdot 3t^2=z(\cos t-6t^2)$，

$$\frac{d^2z}{dt^2}=(\cos t-6t^2)\frac{dz}{dt}+z\frac{d}{dt}(\cos t-6t^2)=e^{\sin t-2t^3}[(\cos t-6t^2)^2-\sin t-12t]。$$

(3) $\frac{dw}{dx}=\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\frac{dy}{dx}+\frac{\partial w}{\partial z}\frac{dz}{dx}$

$$=\frac{ae^{ax}(y-z)}{a^2+1}+\frac{e^{ax}}{a^2+1}\cdot a\cos x-\frac{e^{ax}}{a^2+1}\cdot(-\sin x)$$

$=e^{ax}\sin x$。

(4) $\frac{dz}{dx}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{dv}{dx}=2u\ln v\cdot\frac{1}{y}+\frac{u^2}{v}\cdot 3$

$=\frac{2x}{y^2}\ln(3x-2y)+\frac{3x^2}{y^2(3x-2y)}$，

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=2u\ln v\cdot(-\frac{x}{y^2})+\frac{u^2}{v}\cdot(-2)$

$=-\frac{2x^2}{y^3}\ln(3x-2y)-\frac{2x^2}{y^2(3x-2y)}$。

(5) $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}+\frac{du}{dz}\frac{\partial z}{\partial x}=u\cdot 2x+u\cdot 2z\cdot y^2\cos x$

$=e^{x^2+y^2+y^4\sin^2 x}(2x+2y^4\sin x\cos x)$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y}+\frac{du}{dz}\frac{\partial z}{\partial y}=u\cdot 2y+u\cdot 2z\cdot 2y\sin x$

$=e^{x^2+y^2+y^4\sin^2 x}(2y+4y^3\sin^2 x)$。

(6) 记$u=x^2+y^2+z^2$， $v=x+y+z$。则

$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$

$=x(\sin u+2xv\cos u)+0(\sin u+2yv\cos u)+z(\sin u+2zv\cos u)$

$=te^s(\sin u+2xv\cos u)+e^{s+t}(\sin u+2zv\cos u)$

$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$

$=e^s(\sin u+2xv\cos u)+y(\sin u+2yv\cos u)+z(\sin u+2zv\cos u)$

$=e^s(\sin u+2xv\cos u)+e^t(\sin u+2yv\cos u)+e^{s+t}(\sin u+2zv\cos u)$。

(7) $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=[2x-\sin(x+y)]\cdot 1+[2y-\sin(x+y)]\cdot 0$

$=2(u+v)-\sin(u+v+\arcsin v)$，

$\frac{\partial^2 z}{\partial v\partial u}=\frac{\partial}{\partial v}(\frac{\partial z}{\partial u})=2-\cos(u+v+\arcsin v)(1+\frac{1}{\sqrt{1-v^2}})$。

(8) 记 $v=xy, w=\frac{x}{y}$，则

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=yf_1\left(xy,\frac{x}{y}\right)+\frac{1}{y}f_2\left(xy,\frac{x}{y}\right),$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=xf_1\left(xy,\frac{x}{y}\right)-\frac{x}{y^2}f_2\left(xy,\frac{x}{y}\right),$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})$$

$$=f_1\left(xy,\frac{x}{y}\right)-\frac{1}{y^2}f_2\left(xy,\frac{x}{y}\right)+x\frac{\partial}{\partial x}f_1\left(xy,\frac{x}{y}\right)-\frac{x}{y^2}\frac{\partial}{\partial x}f_2\left(xy,\frac{x}{y}\right)$$

$$=f_1\left(xy,\frac{x}{y}\right)-\frac{1}{y^2}f_2\left(xy,\frac{x}{y}\right)+xyf_{11}\left(xy,\frac{x}{y}\right)-\frac{x}{y^3}f_{22}\left(xy,\frac{x}{y}\right),$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})$$

$$=\frac{2x}{y^3}f_2\left(xy,\frac{x}{y}\right)+x\frac{\partial}{\partial y}f_1\left(xy,\frac{x}{y}\right)-\frac{x}{y^2}\frac{\partial}{\partial y}f_2\left(xy,\frac{x}{y}\right)$$

$$=\frac{2x}{y^3}f_2\left(xy,\frac{x}{y}\right)+x^2f_{11}\left(xy,\frac{x}{y}\right)-\frac{2x^2}{y^2}f_{12}\left(xy,\frac{x}{y}\right)+\frac{x^2}{y^4}f_{22}\left(xy,\frac{x}{y}\right)。$$

(9) 记 $v=x^2+y^2+z^2$，则

$\frac{\partial u}{\partial x}=\frac{df}{dv}\frac{\partial v}{\partial x}=2xf'(x^2+y^2+z^2)$,

$\frac{\partial u}{\partial y}=\frac{df}{dv}\frac{\partial v}{\partial y}=2yf'(x^2+y^2+z^2)$,

$\frac{\partial u}{\partial z}=\frac{df}{dv}\frac{\partial v}{\partial z}=2zf'(x^2+y^2+z^2)$,

$$\frac{\partial^2 u}{\partial x^2} = 2f'(x^2+y^2+z^2) + 2x\frac{\partial}{\partial x}f'(x^2+y^2+z^2)$$

$$= 2f'(x^2+y^2+z^2) + 4x^2 f''(x^2+y^2+z^2),$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2y\frac{\partial}{\partial x}f'(x^2+y^2+z^2)$$

$$= 4xyf''(x^2+y^2+z^2)。$$

(10) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = f_x + f_y + vf_z$,

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = f_x - f_y + uf_z,$$

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\frac{\partial w}{\partial v} = f_z + \frac{\partial f_x}{\partial u} - \frac{\partial f_y}{\partial u} + u\frac{\partial f_z}{\partial u}$$

$$= f_z + \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f_x}{\partial z}\frac{\partial z}{\partial u} - \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial u} - \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial u} - \frac{\partial f_y}{\partial z}\frac{\partial z}{\partial u}$$

$$+ u\left(\frac{\partial f_z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f_z}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f_z}{\partial z}\frac{\partial z}{\partial u}\right)$$

$$= f_{xx} + (u+v)f_{xz} - f_{yy} + (u-v)f_{yz} + f_z + uvf_{zz}。$$

2． 设 $f(x,y)$ 具有连续偏导数，且 $f(x,x^2)=1$， $f_x(x,x^2)=x$，求 $f_y(x,x^2)$。

解 在等式 $f(x,x^2)=1$ 两边对 x 求导，有

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = f_x(x,x^2) + 2xf_y(x,x^2) = 0,$$

再将 $f_x(x,x^2)=x$ 代入，即可得到

$$f_y(x,x^2) = -\frac{1}{2}。$$

3．设 $f(x,y)$ 具有连续偏导数，且 $f(1,1)=1$， $f_x(1,1)=2$， $f_y(1,1)=3$。如果 $\varphi(x) = f(x, f(x,x))$，求 $\varphi'(1)$。

解 $$\frac{d\varphi(x)}{dx} = \frac{\partial f}{\partial x}(x,y(x)) + \frac{\partial f}{\partial y}(x,y(x))\frac{dy(x)}{dx},$$

其中

$$y(x)=f(x,x)\text{，}\quad \frac{dy(x)}{dx}=\frac{\partial f}{\partial x}(x,x)+\frac{\partial f}{\partial y}(x,x)\text{。}$$

由于 $y(1)=f(1,1)=1$，以 $x=1$ 代入上述等式，得到

$$\varphi'(1)=f_x(1,1)+f_y(1,1)(f_x(1,1)+f_y(1,1))=17\text{。}$$

4．设 $z=\dfrac{y}{f(x^2-y^2)}$，其中 $f(t)$ 具有连续导数，且 $f(t)\neq 0$，求 $\dfrac{1}{x}\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\dfrac{\partial z}{\partial y}$。

解. $\dfrac{\partial z}{\partial x}=-\dfrac{y}{f^2(x^2-y^2)}\dfrac{\partial f(x^2-y^2)}{\partial x}=-\dfrac{2xyf'(x^2-y^2)}{f^2(x^2-y^2)}$，

$$\frac{\partial z}{\partial y}=\frac{1}{f(x^2-y^2)}-\frac{y}{f^2(x^2-y^2)}\frac{\partial f(x^2-y^2)}{\partial y}=\frac{1}{f(x^2-y^2)}+\frac{2y^2f'(x^2-y^2)}{f^2(x^2-y^2)}\text{，}$$

直接计算可得

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{yf(x^2-y^2)}\text{。}$$

5．设 $z=\arctan\dfrac{x}{y}$， $x=u+v$， $y=u-v$，验证

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{u-v}{u^2+v^2}\text{。}$$

证 $\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

$$=\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}+\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)=\frac{y-x}{x^2+y^2}\text{，}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}+\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{x}{y^2}=\frac{y+x}{x^2+y^2}\text{，}$$

又由于 $x^2+y^2=(u+v)^2+(u-v)^2=2u^2+2v^2$，所以

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{2y}{x^2+y^2}=\frac{u-v}{u^2+v^2}\text{。}$$

6．设 φ 和 ψ 具有二阶连续导数，验证

（1） $u=y\varphi(x^2-y^2)$ 满足 $y\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial y}=\dfrac{x}{y}u$ ；

（2） $u=\varphi(x-at)+\psi(x+at)$ 满足波动方程 $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$。

证　（1）$\frac{\partial u}{\partial x}=y\frac{\partial\varphi(x^2-y^2)}{\partial x}$

$$=y\varphi'(x^2-y^2)\frac{\partial(x^2-y^2)}{\partial x}=2xy\varphi'(x^2-y^2)\text{，}$$

$$\frac{\partial u}{\partial y}=\varphi(x^2-y^2)+y\frac{\partial\varphi(x^2-y^2)}{\partial y}$$

$$=\varphi(x^2-y^2)+y\varphi'(x^2-y^2)\frac{\partial(x^2-y^2)}{\partial y}=\varphi(x^2-y^2)-2y^2\varphi'(x^2-y^2)\text{，}$$

所以

$$y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=\frac{x}{y}u\text{。}$$

（2）$\frac{\partial u}{\partial x}=\varphi'(x-at)+\psi'(x+at)$，$\frac{\partial^2 u}{\partial x^2}=\varphi''(x-at)+\psi''(x+at)$，

$$\frac{\partial u}{\partial t}=-a\varphi'(x-at)+a\psi'(x+at)\text{，}$$

$$\frac{\partial^2 u}{\partial t^2}=a^2\varphi''(x-at)+a^2\psi''(x+at)\text{，}$$

所以

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}\text{。}$$

7．设 $z=f(x,y)$ 具有二阶连续偏导数，写出 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$ 在坐标变换

$$\begin{cases}u=x^2-y^2,\\ v=2xy\end{cases}$$

下的表达式。

解 $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v}$，

$$\frac{\partial^2 z}{\partial x^2}=2\frac{\partial z}{\partial u}+2x(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\frac{\partial v}{\partial x})+2y(\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x})$$

$$=2\frac{\partial z}{\partial u}+4x^2\frac{\partial^2 z}{\partial u^2}+8xy\frac{\partial^2 z}{\partial v\partial u}+4y^2\frac{\partial^2 z}{\partial v^2}\text{，}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=-2y\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v}\text{，}$$

$$\frac{\partial^2 z}{\partial y^2}=-2\frac{\partial z}{\partial u}-2y(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v\partial u}\frac{\partial v}{\partial y})+2x(\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y})$$

$$=-2\frac{\partial z}{\partial u}+4y^2\frac{\partial^2 z}{\partial u^2}-8xy\frac{\partial^2 z}{\partial v\partial u}+4x^2\frac{\partial^2 z}{\partial v^2}\text{。}$$

由于$u^2+v^2=x^4+2x^2y^2+y^4=(x^2+y^2)^2$，所以

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4(x^2+y^2)(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2})=4\sqrt{u^2+v^2}(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2})$$。

8．设 $f(x,y)=\int_0^{xy}e^{-t^2}dt$，求$\frac{x}{y}\frac{\partial^2 f}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}+\frac{y}{x}\frac{\partial^2 f}{\partial y^2}$。

解 $\frac{\partial f}{\partial x}=ye^{-x^2y^2}$，$\frac{\partial f}{\partial y}=xe^{-x^2y^2}$，

$\frac{\partial^2 f}{\partial x^2}=-2xy^3e^{-x^2y^2}$，$\frac{\partial^2 f}{\partial y\partial x}=e^{-x^2y^2}-2x^2y^2e^{-x^2y^2}$，$\frac{\partial^2 f}{\partial y^2}=-2x^3ye^{-x^2y^2}$。

所以

$$\frac{x}{y}\frac{\partial^2 f}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}+\frac{y}{x}\frac{\partial^2 f}{\partial y^2}=-2e^{-x^2y^2}$$。

9．如果函数$f(x,y)$满足：对于任意的实数t及x,y，成立

$$f(tx,ty)=t^n f(x,y)，$$

那么f称为n次齐次函数。

（1）证明n次齐次函数f满足方程

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$；

（2）利用上述性质，对于$z=\sqrt{x^2+y^2}$求出$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$。

证 在等式$f(tx,ty)=t^n f(x,y)$两边对t求导，

$$\frac{\partial f(tx,ty)}{\partial t}=xf_1(tx,ty)+yf_2(tx,ty)=nt^{n-1}f(x,y)，$$

将t=1 代入即得到

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$。

(2) 由于$z(tx,ty)=tz(x,y)$，所以n=1，由（1）

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\sqrt{x^2+y^2}$$。

10．设$z=f\left(xy,\frac{x}{y}\right)+g\left(\frac{x}{y}\right)$，其中$f$具有二阶连续偏导数，$g$具有二阶

连续导数，求$\dfrac{\partial^2 z}{\partial x\partial y}$。

解　令$u=xy, v=\dfrac{x}{y}$，则

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{dg}{dv}\frac{\partial v}{\partial y}=xf_1(u,v)-\frac{x}{y^2}f_2(u,v)-\frac{x}{y^2}g'(v)，$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1(u,v)-\frac{1}{y^2}f_2(u,v)-\frac{1}{y^2}g'(v)+x(f_{11}(u,v)\frac{\partial u}{\partial x}+f_{12}(u,v)\frac{\partial v}{\partial x})$$

$$-\frac{x}{y^2}[f_{21}(u,v)\frac{\partial u}{\partial x}+f_{22}(u,v)\frac{\partial v}{\partial x}+g''(v)\frac{\partial v}{\partial x}]$$

$$=f_1\left(xy,\frac{x}{y}\right)-\frac{1}{y^2}f_2\left(xy,\frac{x}{y}\right)+xyf_{11}\left(xy,\frac{x}{y}\right)-\frac{x}{y^3}f_{22}\left(xy,\frac{x}{y}\right)$$

$$-\frac{1}{y^2}g'\left(\frac{x}{y}\right)-\frac{x}{y^3}g''\left(\frac{x}{y}\right)。$$

11．设向量值函数$\boldsymbol{f}$：$\mathbf{R}^2\to\mathbf{R}^3$的坐标分量函数为

$$\begin{cases}x=u^2+v^2,\\ y=u^2-v^2,\\ z=uv.\end{cases}$$

向量值函数$\boldsymbol{g}$：$\mathbf{R}^2\to\mathbf{R}^2$的坐标分量函数为

$$\begin{cases}u=r\cos\theta,\\ v=r\sin\theta.\end{cases}$$

求复合函数$\boldsymbol{f}\circ\boldsymbol{g}$的导数。

解　$$\boldsymbol{f}'(u,v)=\begin{pmatrix}2u & 2v\\ 2u & -2v\\ v & u\end{pmatrix}，\quad \boldsymbol{g}'(r,\theta)=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}，$$

所以

$$(\boldsymbol{f}\circ\boldsymbol{g})'(r,\theta)=\boldsymbol{f}'(\boldsymbol{g}(r,\theta))\boldsymbol{g}'(r,\theta)=\begin{pmatrix}2r\cos\theta & 2r\sin\theta\\ 2r\cos\theta & -2r\sin\theta\\ r\sin\theta & r\cos\theta\end{pmatrix}\cdot\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$$

$$=\begin{pmatrix}2r & 0\\ 2r\cos2\theta & -2r^2\sin2\theta\\ r\sin2\theta & r^2\cos2\theta\end{pmatrix}。$$

12．设$w=f(x,u,v)$，$u=g(y,z)$，$v=h(x,y)$，求$\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y},\dfrac{\partial w}{\partial z}$。

解　$$\frac{\partial w}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=f_x+f_v h_x,$$

$$\frac{\partial w}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=f_u g_y+f_v h_y,$$

$$\frac{\partial w}{\partial z}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}=f_u g_z\text{。}$$

13．设$z=u^v$，$u=\ln\sqrt{x^2+y^2}$，$v=\arctan\frac{y}{x}$，求dz。

解　$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=vu^{v-1}\frac{x}{x^2+y^2}+u^v\ln u\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$=vu^{v-1}\frac{x}{x^2+y^2}-u^v\ln u\frac{y}{x^2+y^2},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$=vu^{v-1}\frac{y}{x^2+y^2}+u^v\ln u\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$=vu^{v-1}\frac{y}{x^2+y^2}+u^v\ln u\frac{x}{x^2+y^2},$$

所以

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=u^{v-1}\left(\frac{xdx+ydy}{x^2+y^2}v+\frac{-ydx+xdy}{x^2+y^2}u\ln u\right),$$

其中$u=\ln\sqrt{x^2+y^2}$，$v=\arctan\frac{y}{x}$。

14．设$z=(x^2+y^2)e^{-\arctan\frac{y}{x}}$，求$dz$和$\frac{\partial^2 z}{\partial x\partial y}$。

解　$$\frac{\partial z}{\partial x}=2xe^{-\arctan\frac{y}{x}}+(x^2+y^2)e^{-\arctan\frac{y}{x}}\frac{-1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$=(2x+y)e^{-\arctan\frac{y}{x}},$$

$$\frac{\partial z}{\partial y}=2ye^{-\arctan\frac{y}{x}}+(x^2+y^2)e^{-\arctan\frac{y}{x}}\frac{-1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$=(2y-x)e^{-\arctan\frac{y}{x}},$$

所以

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\left[(2x+y)dx+(2y-x)dy\right]e^{-\arctan\frac{y}{x}}$$；

$$\frac{\partial^2 z}{\partial x\partial y}=e^{-\arctan\frac{y}{x}}+(2x+y)e^{-\arctan\frac{y}{x}}\cdot\frac{-1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{y^2-xy-x^2}{x^2+y^2}e^{-\arctan\frac{y}{x}}$$。

15．求下列函数的全微分：

（1） $u=f(ax^2+by^2+cz^2)$；

（2） $u=f(x+y,xy)$；

（3） $u=f\left(\ln(1+x^2+y^2+z^2),e^{x+y+z}\right)$。

解 (1) 令 $v=ax^2+by^2+cz^2$，则

$$du=f'(v)(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz)$$

$$=2f'(ax^2+by^2+cz^2)(axdx+bydy+czdz)$$。

(2) $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$

$$=(f_1+yf_2)dx+(f_1+xf_2)dy$$。

(3) $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$

$$=\frac{2f_1}{1+x^2+y^2+z^2}(xdx+ydy+zdz)+(e^{x+y+z}f_2)(dx+dy+dz)$$。

16．设 $f(t)$ 具有任意阶连续导数，而 $u=f(ax+by+cz)$。对任意正整数 k，求 $\mathrm{d}^k u$。

解　当 $k=1$ 时，成立

$$du=f'(ax+by+cz)d(ax+by+cz)=f'(ax+by+cz)(adx+bdy+cdz)$$，

应用数学归纳法，假设对于 k 成立

$$d^k u=f^{(k)}(ax+by+cz)(adx+bdy+cdz)^k$$，

则对于 $k+1$ 成立

$$d^{k+1}u=d(d^k u)=d[f^{(k)}(ax+by+cz)(adx+bdy+cdz)^k]$$

$$=f^{(k+1)}(ax+by+cz)(adx+bdy+cdz)^{k+1}$$。

由数学归纳法可知对任意正整数 k 成立

$$d^k u = f^{(k)}(ax+by+cz)(adx+bdy+cdz)^k。$$

17. 设函数 $z = f(x,y)$ 在全平面上有定义，具有连续的偏导数，且满足方程

$$xf_x(x,y)+yf_y(x,y)=0,$$

证明：$f(x,y)$ 为常数。

证 当 $r \neq 0$ 时，

$$\frac{\partial}{\partial r} f(r\cos\theta, r\sin\theta) = \cos\theta\, f_x(r\cos\theta, r\sin\theta) + \sin\theta\, f_y(r\cos\theta, r\sin\theta)$$

$$= \frac{1}{r}\left(xf_x(x,y)+yf_y(x,y)\right)=0,$$

所以

$$f(r\cos\theta, r\sin\theta) = F(\theta)。$$

再利用 $f(x,y)$ 在 $(0,0)$ 点的连续性，得到

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = F(\theta) = f(0,0),$$

即 $F(\theta)$ 为常数，所以 $f(x,y)$ 为常数。

18. 设 n 元函数 f 在 $\mathbf{R}^n$ 上具有连续偏导数，证明对于任意的 $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$，$\boldsymbol{y}=(y_1,y_2,\cdots,y_n)\in\mathbf{R}^n$，成立下述 **Hadamard** 公式：

$$f(\mathbf{y})-f(\mathbf{x})=\sum_{i=1}^{n}\int_0^1 (y_i-x_i)\frac{\partial f}{\partial x_i}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))dt。$$

证 设 $F(t)=f\left(\mathsf{x}+t(\mathsf{y}-\mathsf{x})\right)$，则

$$f(\mathbf{y})-f(\mathbf{x})=F(1)-F(0)=\int_0^1 F'(t)dt。$$

由于

$$F'(t)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\boldsymbol{x}+t(\boldsymbol{y}-\boldsymbol{x}))\frac{\partial(x_i+t(y_i-x_i))}{\partial t}$$

$$=\sum_{i=1}^{n}(y_i-x_i)\frac{\partial f}{\partial x_i}(\boldsymbol{x}+t(\boldsymbol{y}-\boldsymbol{x}))。$$

所以

$$f(\mathbf{y})-f(\mathbf{x})=F(1)-F(0)$$

$$=\sum_{i=1}^{n}\int_0^1 (y_i-x_i)\frac{\partial f}{\partial x_i}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))dt。$$