习　题　12.3　Taylor 公式

1. 对函数 $f(x,y)=\sin x\cos y$ 应用中值定理证明：存在 $\theta\in(0,1)$，使得

$$\frac{3}{4}=\frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}。$$

证 设 $(x_0,y_0)=(0,0)$, $(\Delta x,\Delta y)=(\frac{\pi}{3},\frac{\pi}{6})$，对函数 $f(x,y)=\sin x\cos y$ 应用微分中值定理（即 $k=0$ 时的 Taylor 公式），可知存在 $\theta\in(0,1)$，使得

$$\frac{3}{4}=f(\frac{\pi}{3},\frac{\pi}{6})-f(0,0)=f_x(\theta\Delta x,\theta\Delta y)\Delta x+f_y(\theta\Delta x,\theta\Delta y)\Delta y$$

$$=\frac{\pi}{3}\cos\frac{\pi\theta}{3}\cos\frac{\pi\theta}{6}-\frac{\pi}{6}\sin\frac{\pi\theta}{3}\sin\frac{\pi\theta}{6}。$$

2. 写出函数 $f(x,y)=3x^3+y^3-2x^2y-2xy^2-6x-8y+9$ 在点 $(1,2)$ 的 Taylor 展开式。

解 $$f(x,y)=3[(x-1)+1]^3+[(y-2)+2]^3-2[(x-1)+1]^2[(y-2)+2]$$

$$-2[(x-1)+1][(y-2)+2]^2-6[(x-1)+1]-8[(y-2)+2]+9$$

$$=-14-13(x-1)-6(y-2)+5(x-1)^2-12(x-1)(y-2)+4(y-2)^2$$

$$+3(x-1)^3-2(x-1)^2(y-2)-2(x-1)(y-2)^2+(y-2)^3。$$

注　本题也可设 $u=x-1, v=y-2$，于是

$$f(x,y)=f(u+1,v+2)$$

$$=3(u+1)^3+(v+2)^3-2(u+1)^2(v+2)-2(u+1)(v+2)^2-6(u+1)-8(v+2)+9,$$

展开后再用 $u=x-1, v=y-2$ 代换回来。

3. 求函数 $f(x,y)=\sin x\ln(1+y)$ 在 $(0,0)$ 点的 Taylor 展开式（展开到三阶导数为止）。

解 $$f(x,y)=(x-\frac{x^3}{6}+o(x^3))(y-\frac{y^2}{2}+\frac{y^3}{3}+o(y^3))$$

$$=xy-\frac{1}{2}xy^2+o\left((\sqrt{x^2+y^2})^3\right)。$$

4. 求函数 $f(x,y)=e^{x+y}$ 在 $(0,0)$ 点的 n 阶 Taylor 展开式，并写出余项。

解 $$f(x,y)=1+(x+y)+\frac{1}{2!}(x+y)^2+\cdots+\frac{1}{n!}(x+y)^n+R_n,$$

其中$R_n=\frac{1}{(n+1)!}(x+y)^{n+1}e^{\theta(x+y)}$。

5. 设$f(x,y)=\frac{\cos y}{x},\quad x>0$。

（1）求$f(x,y)$在$(1,0)$点的 Taylor 展开式（展开到二阶导数），并计算余项R_2；

（2）求$f(x,y)$在$(1,0)$点的k阶 Taylor 展开式，并证明在$(1,0)$点的某个领域内，余项R_k满足当$k\to\infty$时，$R_k\to 0$。

解 （1） $f(x,y)=1-(x-1)+(x-1)^2-\frac{1}{2}y^2+R_2$，

$$R_2=\frac{1}{3!}\left[(x-1)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]^3 f(1+\theta(x-1),\theta y)$$

$$=-\frac{\cos\eta}{\xi^4}(x-1)^3-\frac{\sin\eta}{\xi^3}(x-1)^2y+\frac{\cos\eta}{2\xi^2}(x-1)y^2+\frac{\sin\eta}{6\xi}y^3，$$

其中$\xi=1+\theta(x-1)$，$\eta=\theta y$，$0<\theta<1$。

（2） $$f(x,y)=1+\sum_{n=1}^{k}\left[\frac{1}{n!}\sum_{j=0}^{n}C_n^j(-1)^{n-j}(n-j)!\cos(\frac{j}{2}\pi)(x-1)^{n-j}y^j\right]+R_k，$$

$$R_k=\frac{1}{(k+1)!}\sum_{j=0}^{k+1}C_{k+1}^j(-1)^{k+1-j}(k+1-j)!\frac{1}{\xi^{k-j+2}}\cos(\eta+\frac{j}{2}\pi)(x-1)^{k+1-j}y^j。$$

当$x=1$时，$\xi=1$，对任意$y\in(-\infty,+\infty)$，$R_k\to 0\ (k\to\infty)$显然成立；

当$0<|x-1|<\frac{1}{3}$时，$\frac{2}{3}<\xi<\frac{4}{3}$，$\left|\frac{x-1}{\xi}\right|<\frac{1}{2}$，于是对任意$y\in(-\infty,+\infty)$，有

$$|R_k|\le\frac{1}{(k+1)!}\sum_{j=0}^{k+1}\frac{(k+1)!}{j!(k+1-j)!}(k+1-j)!\frac{1}{|\xi|^{k-j+2}}|x-1|^{k+1-j}|y|^j$$

$$=\frac{1}{|\xi|}\sum_{j=0}^{k+1}\frac{1}{j!}\left|\frac{x-1}{\xi}\right|^{k+1-j}|y|^j\le\frac{1}{|\xi|}\left|\frac{x-1}{\xi}\right|^{k+1}\sum_{j=0}^{\infty}\frac{1}{j!}\left|\frac{y\xi}{x-1}\right|^j=\frac{1}{|\xi|}\left|\frac{x-1}{\xi}\right|^{k+1}e^{\left|\frac{y\xi}{x-1}\right|}，$$

因此也成立$R_k\to 0$ （$k\to\infty$）。

6. 利用 Taylor 公式近似计算$8.96^{2.03}$ （展开到二阶导数）。

解 考虑$f(x,y)=(9+x)^{2+y}$在$(0,0)$点的 Taylor 公式:

$$f(x,y)=81+18x+81\ln 9\,y+x^2+(9+18\ln 9)xy+\frac{81}{2}\ln^2 9\,y^2+R_2(x,y)，$$

于是

$$8.96^{2.03}=f(-0.04,0.03)\approx 81+18(-0.04)+81\ln 9\cdot 0.03$$

$+(-0.04)^2+(9+18\ln 9)\cdot(-0.04)\cdot 0.03+81/2\cdot\ln^2(9)\cdot 0.03^2$

≈ 85.74。

7．设 $f(x,y)$ 在 $\mathbf{R}^2$ 上可微。$\boldsymbol{l}_1$ 与 $\boldsymbol{l}_2$ 是 $\mathbf{R}^2$ 上两个线性无关的单位向量（方向）。若

$$\frac{\partial f}{\partial l_i}(x,y)\equiv 0\text{，}\quad i=1,2\text{，}$$

证明：在 $\mathbf{R}^2$ 上 $f(x,y)\equiv$ 常数。

证 设 $\boldsymbol{l}_1=(\cos\alpha_1,\sin\alpha_1)$，$\boldsymbol{l}_2=(\cos\alpha_2,\sin\alpha_2)$。由于 $f(x,y)$ 在 $\mathbf{R}^2$ 上可微，

$$\frac{\partial f}{\partial l_1}(x,y)=f_x(x,y)\cos\alpha_1+f_y(x,y)\sin\alpha_1\equiv 0\text{，}$$

$$\frac{\partial f}{\partial l_2}(x,y)=f_x(x,y)\cos\alpha_2+f_y(x,y)\sin\alpha_2\equiv 0\text{。}$$

因为 $\boldsymbol{l}_1$ 与 $\boldsymbol{l}_2$ 线性无关，所以

$$\begin{vmatrix}\cos\alpha_1 & \sin\alpha_1\\ \cos\alpha_2 & \sin\alpha_2\end{vmatrix}\neq 0\text{，}$$

因此上面的线性方程组只有零解，即

$$f_x(x,y)\equiv 0\text{，}\quad f_y(x,y)\equiv 0\text{。}$$

于是由推论 12.3.1 知道 $f(x,y)\equiv$ 常数。

8．设 $f(x,y)=\sin\dfrac{y}{x}$（$x\neq 0$），证明：

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^k f(x,y)\equiv 0\text{，}\quad k\geq 1\text{。}$$

证 因为

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(x,y)=x\cdot\cos\frac{y}{x}\cdot\left(-\frac{y}{x^2}\right)+y\cdot\cos\frac{y}{x}\cdot\frac{1}{x}\equiv 0,$$

所以当 $k>1$ 时成立

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^k f(x,y)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{k-1}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(x,y)\equiv 0\text{。}$$

习　题 12.4　隐函数

1.　　求下列方程所确定的隐函数的导数或偏导数：

(1) $\sin y+e^{x}-xy^{2}=0$，求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$；

(2) $x^{y}=y^{x}$，求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$；

(3) $\ln\sqrt{x^{2}+y^{2}}=\arctan\dfrac{y}{x}$，求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$；

(4) $\arctan\dfrac{x+y}{a}-\dfrac{y}{a}=0$，求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$和$\dfrac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}$；

(5) $\dfrac{x}{z}=\ln\dfrac{z}{y}$，求$\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$；

(6) $e^{z}-xyz=0$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$，$\dfrac{\partial^{2}z}{\partial x^{2}}$和$\dfrac{\partial^{2}z}{\partial x\partial y}$；

(7) $z^{3}-3xyz=a^{3}$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$，$\dfrac{\partial^{2}z}{\partial x^{2}}$和$\dfrac{\partial^{2}z}{\partial x\partial y}$；

(8) $f(x+y,y+z,z+x)=0$，求$\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$；

(9) $z=f(xz,z-y)$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$和$\dfrac{\partial^{2}z}{\partial x^{2}}$；

(10) $f(x,x+y,x+y+z)=0$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$，$\dfrac{\partial^{2}z}{\partial x^{2}}$和$\dfrac{\partial^{2}z}{\partial x\partial y}$。

解　(1) 设$F(x,y)=\sin y+e^{x}-xy^{2}=0$，则

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{y^{2}-e^{x}}{\cos y-2xy}。$$

(2) 设$F(x,y)=x^{y}-y^{x}=0$，则

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{y(x\ln y-y)}{x(y\ln x-x)}。$$

注　本题也可先在等式$x^{y}=y^{x}$两边取对数，然后设

$$G(x,y)=y\ln x-x\ln y=0。$$

(3) 设$F(x,y)=\ln\sqrt{x^{2}+y^{2}}-\arctan\dfrac{y}{x}=0$，则

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{x+y}{x-y}。$$

(4) 设 $F(x,y)=\arctan\frac{x+y}{a}-\frac{y}{a}=0$，则

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{a^2}{(x+y)^2}\text{，}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{2a^2}{(x+y)^3}\left(1+\frac{dy}{dx}\right)=-\frac{2a^2}{(x+y)^5}[a^2+(x+y)^2]\text{。}$$

(5) 设 $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}=0$，则

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{z}{x+z}\text{，}\quad\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{z^2}{y(x+z)}\text{。}$$

(6) 设 $F(x,y,z)=e^z-xyz=0$，则

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{yz}{e^z-xy}\text{，}\quad\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xz}{e^z-xy}\text{，}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{y}{e^z-xy}\cdot\frac{\partial z}{\partial x}-\frac{yz}{(e^z-xy)^2}\left(e^z\frac{\partial z}{\partial x}-y\right)=\frac{2y^2z}{(e^z-xy)^2}-\frac{y^2z^2e^z}{(e^z-xy)^3}\text{，}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{1}{e^z-xy}\left(z+x\frac{\partial z}{\partial x}\right)-\frac{xz}{(e^z-xy)^2}\left(e^z\frac{\partial z}{\partial x}-y\right)$$

$$=\frac{z}{e^z-xy}+\frac{2xyz}{(e^z-xy)^2}-\frac{xyz^2e^z}{(e^z-xy)^3}\text{。}$$

(7) 设 $F(x,y,z)=z^3-3xyz-a^3=0$，则

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{yz}{z^2-xy}\text{，}\quad\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xz}{z^2-xy}\text{，}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{y}{z^2-xy}\left(\frac{\partial z}{\partial x}\right)-\frac{yz}{(z^2-xy)^2}\left(2z\frac{\partial z}{\partial x}-y\right)=-\frac{2xy^3z}{(z^2-xy)^3}\text{，}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{1}{z^2-xy}\left(z+x\frac{\partial z}{\partial x}\right)-\frac{xz}{(z^2-xy)^2}\left(2z\frac{\partial z}{\partial x}-y\right)$$

$$=\frac{z^5-2xyz^3-x^2y^2z}{(z^2-xy)^3}\text{。}$$

(8) 由 $f(x+y,y+z,z+x)=0$ 即可得到

$$\frac{\partial z}{\partial x}=-\frac{f_1+f_3}{f_2+f_3}，\ \frac{\partial z}{\partial y}=-\frac{f_1+f_2}{f_2+f_3}。$$

(9) 设 $F(x,y,z)=z-f(xz,z-y)=0$，则

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{zf_1}{1-xf_1-f_2}，\ \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{f_2}{1-xf_1-f_2}，$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{1}{1-xf_1-f_2}\left[\frac{\partial z}{\partial x}f_1+z\left(z+x\frac{\partial z}{\partial x}\right)f_{11}+z\frac{\partial z}{\partial x}f_{12}\right]$$

$$+\frac{zf_1}{(1-xf_1-f_2)^2}\left[f_1+x\left(z+x\frac{\partial z}{\partial x}\right)f_{11}+x\frac{\partial z}{\partial x}f_{12}+\left(z+x\frac{\partial z}{\partial x}\right)f_{21}+\frac{\partial z}{\partial x}f_{22}\right]$$

$$=\frac{1}{1-xf_1-f_2}\left[2\frac{\partial z}{\partial x}f_1+\left(z+x\frac{\partial z}{\partial x}\right)^2f_{11}+2\frac{\partial z}{\partial x}\left(z+x\frac{\partial z}{\partial x}\right)f_{12}+\left(\frac{\partial z}{\partial x}\right)^2f_{22}\right]。$$

(10) 由 $f(x,x+y,x+y+z)=0$ 即可得到

$$\frac{\partial z}{\partial x}=-\frac{f_1+f_2+f_3}{f_3}，\ \frac{\partial z}{\partial y}=-\frac{f_2+f_3}{f_3}，$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{1}{f_3}\left[f_{11}+f_{12}+\left(1+\frac{\partial z}{\partial x}\right)f_{13}+f_{21}+f_{22}+\left(1+\frac{\partial z}{\partial x}\right)f_{23}\right]$$

$$+\frac{f_1+f_2}{f_3}\left[f_{31}+f_{32}+\left(1+\frac{\partial z}{\partial x}\right)f_{33}\right]$$

$$=-\frac{1}{f_3^3}\left[f_3^2(f_{11}+2f_{12}+f_{22})-2f_3(f_1+f_2)(f_{13}+f_{23})+(f_1+f_2)^2f_{33}\right]，$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=-\frac{1}{f_3}\left[f_{21}+f_{22}+\left(1+\frac{\partial z}{\partial x}\right)f_{23}\right]+\frac{f_2}{f_3^2}\left[f_{31}+f_{32}+\left(1+\frac{\partial z}{\partial x}\right)f_{33}\right]$$

$$=-\frac{1}{f_3^3}\left[f_3^2(f_{12}+f_{22})-f_2f_3f_{13}+f_2(f_1+f_2)f_{33}-f_3(f_1+2f_2)f_{23}\right]。$$

2．设 $y=\tan(x+y)$ 确定 y 为 x 的隐函数，验证

$$\frac{d^3y}{dx^3}=-\frac{2(3y^4+8y^2+5)}{y^8}。$$

证　由

$$y'=\sec^2(x+y)(1+y')=(1+y^2)(1+y')$$

解出

$$y'=-1-\frac{1}{y^2},$$

再求二阶和三阶导数，有

$$y''=\frac{2}{y^3}y'=-\frac{2}{y^3}-\frac{2}{y^5},$$

$$y'''=\left(\frac{6}{y^4}+\frac{10}{y^6}\right)y'=-\frac{2(3y^4+8y^2+5)}{y^8}。$$

3．设ϕ是可微函数，证明由$\phi(cx-az,cy-bz)=0$所确定的隐函数$z=f(x,y)$满足方程

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c。$$

证　由$\phi(cx-az,cy-bz)=0$可得到

$$\frac{\partial z}{\partial x}=-\frac{c\phi_1}{-a\phi_1-b\phi_2}=\frac{c\phi_1}{a\phi_1+b\phi_2},\quad \frac{\partial z}{\partial y}=-\frac{c\phi_2}{-a\phi_1-b\phi_2}=\frac{c\phi_2}{a\phi_1+b\phi_2},$$

所以

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c。$$

4．设方程$\phi(x+zy^{-1},y+zx^{-1})=0$确定隐函数$z=f(x,y)$，证明它满足方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy。$$

证　由于

$$\frac{\partial z}{\partial x}=-\frac{\phi_1+z\left(-\frac{1}{x^2}\right)\phi_2}{\frac{1}{y}\phi_1+\frac{1}{x}\phi_2}=\frac{yz\phi_2-x^2y\phi_1}{x(x\phi_1+y\phi_2)},\quad \frac{\partial z}{\partial y}=-\frac{z\left(-\frac{1}{y^2}\right)\phi_1+\phi_2}{\frac{1}{y}\phi_1+\frac{1}{x}\phi_2}=\frac{xz\phi_1-xy^2\phi_2}{y(x\phi_1+y\phi_2)},$$

所以

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy。$$

5．求下列方程组所确定的隐函数的导数或偏导数：

（1）$\begin{cases}z-x^2-y^2=0,\\ x^2+2y^2+3z^2=4a^2,\end{cases}$ 求$\frac{\mathrm{d}y}{\mathrm{d}x}$，$\frac{\mathrm{d}z}{\mathrm{d}x}$，$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$和$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$；

（2）$\begin{cases}xu+yv=0,\\ yu+xv=1,\end{cases}$ 求$\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial^2u}{\partial x^2}$和$\frac{\partial^2u}{\partial x\partial y}$；

（3）$\begin{cases}u=f(ux,v+y),\\ v=g(u-x,v^2y),\end{cases}$ 求$\frac{\partial u}{\partial x}$和$\frac{\partial v}{\partial x}$；

（4）$\begin{cases}x=u+v,\\ y=u-v,\\ z=u^2v^2,\end{cases}$ 求$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$；

(5) $\begin{cases} x = e^u \cos v, \\ y = e^u \sin v, \\ z = u^2 + v^2, \end{cases}$ 求$\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$。

解 (1) 在方程组中对 x 求导，得到

$$\begin{cases} \dfrac{dz}{dx} - 2x - 2y\dfrac{dy}{dx} = 0, \\ 2x + 4y\dfrac{dy}{dx} + 6z\dfrac{dz}{dx} = 0, \end{cases}$$

由此解出

$$\frac{dy}{dx} = -\frac{x(1+6z)}{y(2+6z)}，\quad \frac{dz}{dx} = \frac{x}{1+3z}。$$

再求二阶导数，得到

$$\frac{d^2y}{dx^2} = -\frac{(1+6z)}{y(2+6z)} + \frac{x(1+6z)}{y^2(2+6z)}\frac{dy}{dx} + \frac{x(-3)}{2y(1+3z)^2}\frac{dz}{dx}$$

$$= \frac{1}{2y}\left[\frac{1}{1+3z} - \frac{x^2(1+6z)^2}{2y^2(1+3z)^2} - \frac{3x^2}{(1+3z)^3} - 2\right],$$

$$\frac{d^2z}{dx^2} = \frac{1}{1+3z} - \frac{3x}{(1+3z)^2}\frac{dz}{dx} = \frac{1}{1+3z} - \frac{3x^2}{(1+3z)^3}。$$

(2) 在方程组中对 x 求偏导，得到

$$\begin{cases} u + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial v}{\partial x} = 0, \\ y\dfrac{\partial u}{\partial x} + v + x\dfrac{\partial v}{\partial x} = 0, \end{cases}$$

解此方程组，得到

$$\frac{\partial u}{\partial x} = \frac{ux - vy}{y^2 - x^2}，\quad \frac{\partial v}{\partial x} = \frac{vx - uy}{y^2 - x^2}。$$

在方程组中对 y 求偏导，得到

$$\begin{cases} x\dfrac{\partial u}{\partial y}+v+y\dfrac{\partial v}{\partial y}=0, \\ u+y\dfrac{\partial u}{\partial y}+x\dfrac{\partial v}{\partial y}=0, \end{cases}$$

解此方程组，得到

$$\frac{\partial u}{\partial y}=\frac{vx-uy}{y^2-x^2}，\ \frac{\partial v}{\partial y}=\frac{ux-vy}{y^2-x^2}。$$

于是

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{y^2-x^2}\left(u+x\frac{\partial u}{\partial x}-y\frac{\partial v}{\partial x}\right)+\frac{ux-vy}{(y^2-x^2)^2}2x=\frac{2u(x^2+y^2)-4xyv}{(y^2-x^2)^2},$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{y^2-x^2}\left(v+x\frac{\partial v}{\partial x}-y\frac{\partial u}{\partial x}\right)+\frac{vx-uy}{(y^2-x^2)^2}2x=\frac{2v(x^2+y^2)-4xyu}{(y^2-x^2)^2}。$$

(3) 在方程组中对 x 求偏导，得到

$$\begin{cases} \dfrac{\partial u}{\partial x}=\left(u+x\dfrac{\partial u}{\partial x}\right)f_1+\dfrac{\partial v}{\partial x}f_2, \\ \dfrac{\partial v}{\partial x}=\left(\dfrac{\partial u}{\partial x}-1\right)g_1+2vyg_2\dfrac{\partial v}{\partial x}, \end{cases}$$

解此方程组，得到

$$\frac{\partial u}{\partial x}=\frac{f_2g_1+uf_1(2vyg_2-1)}{f_2g_1-(xf_1-1)(2vyg_2-1)}，\ \frac{\partial v}{\partial x}=\frac{(1-xf_1)g_1-uf_1g_1}{f_2g_1-(xf_1-1)(2vyg_2-1)}。$$

(4) 在方程组中分别对 x 与 y 求偏导，得到

$$\begin{cases} 1=\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x}, \\ 0=\dfrac{\partial u}{\partial x}-\dfrac{\partial v}{\partial x}, \end{cases} \text{与} \begin{cases} 0=\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial y}, \\ 1=\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial y}, \end{cases}，$$

解此两方程组，得到

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{1}{2} \text{ 与 } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial y}=\frac{1}{2},$$

所以

$$\frac{\partial z}{\partial x}=2uv^2\frac{\partial u}{\partial x}+2u^2v\frac{\partial v}{\partial x}=uv(u+v),$$

$$\frac{\partial z}{\partial y}=2uv^2\frac{\partial u}{\partial y}+2u^2v\frac{\partial v}{\partial y}=uv(u-v)\text{。}$$

(5) 在方程组中对 x 求偏导，得到

$$\begin{cases}1=e^u\cos v\dfrac{\partial u}{\partial x}-e^u\sin v\dfrac{\partial v}{\partial x},\\ 0=e^u\sin v\dfrac{\partial u}{\partial x}+e^u\cos v\dfrac{\partial v}{\partial x},\end{cases}$$ ，

解此方程组，得到

$$\frac{\partial u}{\partial x}=e^{-u}\cos v,\frac{\partial v}{\partial x}=-e^{-u}\sin v\text{，}$$

所以

$$\frac{\partial z}{\partial x}=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=\frac{2(u\cos v-v\sin v)}{e^u}\text{。}$$

在方程组中对 y 求偏导，得到

$$\begin{cases}0=e^u\cos v\dfrac{\partial u}{\partial y}-e^u\sin v\dfrac{\partial v}{\partial y},\\ 1=e^u\sin v\dfrac{\partial u}{\partial y}+e^u\cos v\dfrac{\partial v}{\partial y},\end{cases}$$ ，

解此方程组，得到

$$\frac{\partial u}{\partial x}=e^{-u}\sin v,\frac{\partial v}{\partial x}=e^{-u}\cos v\text{，}$$

所以

$$\frac{\partial z}{\partial y}=2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=\frac{2(v\cos v+u\sin v)}{e^u}\text{。}$$

6．求微分

（1） $x+2y+z-2\sqrt{xyz}=0$，求 $\mathrm{d}z$ ；

（2） $\begin{cases}x+y=u+v,\\ \dfrac{x}{y}=\dfrac{\sin u}{\sin v},\end{cases}$ 求 $\mathrm{d}u$ 与 $\mathrm{d}v$。

解 (1) 直接对等式两边求微分，得到

$$dx+2dy+dz-\frac{1}{\sqrt{xyz}}(yzdx+xzdy+xydz)=0\text{，}$$

由此解出

$$dz=\frac{yz-\sqrt{xyz}}{\sqrt{xyz}-xy}dx+\frac{xz-2\sqrt{xyz}}{\sqrt{xyz}-xy}dy\text{。}$$

(2) 直接在方程组中求微分，得到

$$\begin{cases}dx+dy=du+dv,\\ \dfrac{1}{y}dx-\dfrac{x}{y^2}dy=\dfrac{\cos u}{\sin v}du-\dfrac{\sin u\cos v}{\sin^2 v}dv,\end{cases}$$

解此方程组，得到

$$du=\frac{\sin v+x\cos v}{x\cos v+y\cos u}dx+\frac{x\cos v-\sin u}{x\cos v+y\cos u}dy\,,$$

$$dv=\frac{y\cos u-\sin v}{x\cos v+y\cos u}dx+\frac{y\cos u+\sin u}{x\cos v+y\cos u}dy\text{。}$$

7． 设$\begin{cases}x=x(y),\\ z=z(y)\end{cases}$是由方程组$\begin{cases}F(y-x,\,y-z)=0,\\ G\left(xy,\dfrac{z}{y}\right)=0\end{cases}$所确定的向量值隐函数，其中二元函数$F$和$G$分别具有连续的偏导数，求$\dfrac{dx}{dy}$和$\dfrac{dz}{dy}$。

解 (1) 在方程组中对y求导数，有

$$\begin{cases}\left(1-\dfrac{dx}{dy}\right)F_1+\left(1-\dfrac{dz}{dy}\right)F_2=0,\\ \left(x+y\dfrac{dx}{dy}\right)G_1+\left(-\dfrac{z}{y^2}+\dfrac{1}{y}\dfrac{dz}{dy}\right)G_2=0,\end{cases}\,,$$

解此方程组，得到

$$\frac{dx}{dy}=\frac{yF_1G_2+xy^2F_2G_1+(y-z)F_2G_2}{y(F_1G_2-y^2F_2G_1)}\,,$$

$$\frac{dz}{dy}=\frac{zF_1G_2-y^3F_2G_1-y^2(x+y)F_1G_1}{y(F_1G_2-y^2F_2G_1)}\text{。}$$

8．设$f(x,y)$具有二阶连续偏导数。在极坐标$\begin{cases}x=r\cos\theta,\\ y=r\sin\theta\end{cases}$变换下，求

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

关于极坐标的表达式。

解　经计算，有

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}=\cos\theta\frac{\partial f}{\partial x}+\sin\theta\frac{\partial f}{\partial y},$$

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial f}{\partial x}+r\cos\theta\frac{\partial f}{\partial y},$$

$$\frac{\partial^2 f}{\partial r^2}=\cos\theta\left(\cos\theta\frac{\partial^2 f}{\partial x^2}-\sin\theta\frac{\partial^2 f}{\partial y\partial x}\right)-\sin\theta\left(\cos\theta\frac{\partial^2 f}{\partial x\partial y}-\sin\theta\frac{\partial^2 f}{\partial y^2}\right)$$

$$=\cos^2\theta\frac{\partial^2 f}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial^2 f}{\partial y\partial x}+\sin^2\theta\frac{\partial^2 f}{\partial y^2},$$

$$\frac{\partial^2 f}{\partial \theta^2}=-r\cos\theta\frac{\partial f}{\partial x}-r\sin\theta\frac{\partial f}{\partial y}-r\sin\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x^2}+r\cos\theta\frac{\partial^2 f}{\partial y\partial x}\right)$$

$$+r\cos\theta\left(-r\sin\theta\frac{\partial^2 f}{\partial x\partial y}+r\cos\theta\frac{\partial^2 f}{\partial^2 y}\right)$$

$$=-r\cos\theta\frac{\partial f}{\partial x}-r\sin\theta\frac{\partial f}{\partial y}+r^2\left(\sin^2\theta\frac{\partial^2 f}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 f}{\partial y\partial x}+\cos^2\theta\frac{\partial^2 f}{\partial y^2}\right),$$

容易验证

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{1}{r}\frac{\partial f}{\partial r}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}。$$

9．设二元函数 f 具有二阶连续偏导数。证明：通过适当线性变换

$$\begin{cases}u=x+\lambda y,\\ v=x+\mu y,\end{cases}$$

可以将方程

$$A\frac{\partial^2 f}{\partial x^2}+2B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}=0\qquad（AC-B^2<0）.$$

化简为

$$\frac{\partial^2 f}{\partial u\partial v}=0。$$

并说明此时 λ,μ 为一元二次方程 $A+2Bt+Ct^2=0$ 的两个相异实根。

证　经计算，有

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v},$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\lambda\frac{\partial f}{\partial u}+\mu\frac{\partial f}{\partial v},$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial v\partial u}\frac{\partial v}{\partial x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x}=\frac{\partial^2 f}{\partial u^2}+2\frac{\partial^2 f}{\partial v\partial u}+\frac{\partial^2 f}{\partial v^2}\text{，}$$

$$\frac{\partial^2 f}{\partial y^2}=\lambda\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial v\partial u}\frac{\partial v}{\partial y}\right)+\mu\left(\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)$$

$$=\lambda^2\frac{\partial^2 f}{\partial u^2}+2\lambda\mu\frac{\partial^2 f}{\partial v\partial u}+\mu^2\frac{\partial^2 f}{\partial y^2}\text{，}$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial v\partial u}\frac{\partial v}{\partial y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}=\lambda\frac{\partial^2 f}{\partial u^2}+(\lambda+\mu)\frac{\partial^2 f}{\partial v\partial u}+\mu\frac{\partial^2 f}{\partial v^2}\text{，}$$

所以

$$0=A\frac{\partial^2 f}{\partial x^2}+2B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}$$

$$=(A+2B\lambda+C\lambda^2)\frac{\partial^2 f}{\partial u^2}+(A+2B\mu+C\mu^2)\frac{\partial^2 f}{\partial v^2}+2[A+B(\lambda+\mu)+C\lambda\mu]\frac{\partial^2 f}{\partial v\partial u}\text{。}$$

由条件 $AC-B^2<0$ 知一元二次方程 $A+2Bt+Ct^2=0$ 有两个相异实根，所以只要取 λ,μ 为方程的两个相异实根。此时由 $\lambda+\mu=-\frac{2B}{C}$ 与 $\lambda\mu=\frac{A}{C}$，可得

$$A+B(\lambda+\mu)+C\lambda\mu=2\frac{AC-B^2}{C}\neq 0\text{，}$$

于是原方程化简为 $\frac{\partial^2 f}{\partial u\partial v}=0$。

10． 通过自变量变换 $\begin{cases}x=\mathrm{e}^{\xi},\\ y=\mathrm{e}^{\eta}\end{cases}$ 变换方程

$$ax^2\frac{\partial^2 z}{\partial x^2}+2bxy\frac{\partial^2 z}{\partial x\partial y}+cy^2\frac{\partial^2 z}{\partial y^2}=0\text{，}\quad a,b,c\text{ 为常数。}$$

解 由 $\xi=\ln x,\eta=\ln y$，可得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x}=\frac{1}{x}\frac{\partial z}{\partial \xi}\text{，}\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y}=\frac{1}{y}\frac{\partial z}{\partial \eta}\text{，}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{x^2}\frac{\partial^2 z}{\partial \xi^2}-\frac{1}{x^2}\frac{\partial z}{\partial \xi}\text{，}\quad \frac{\partial^2 z}{\partial y^2}=\frac{1}{y^2}\frac{\partial^2 z}{\partial \eta^2}-\frac{1}{y^2}\frac{\partial z}{\partial \eta}\text{，}\quad \frac{\partial^2 z}{\partial y\partial x}=\frac{1}{xy}\frac{\partial^2 z}{\partial \eta\partial \xi}\text{。}$$

代入原方程，得到

$$ax^2\frac{\partial^2 z}{\partial x^2}+2bxy\frac{\partial^2 z}{\partial x\partial y}+cy^2\frac{\partial^2 z}{\partial y^2}=a\left(\frac{\partial^2 z}{\partial \xi^2}-\frac{\partial z}{\partial \xi}\right)+2b\frac{\partial^2 z}{\partial \xi\partial \eta}+c\left(\frac{\partial^2 z}{\partial \eta^2}-\frac{\partial z}{\partial \eta}\right)=0。$$

11．通过自变量变换$\begin{cases}u=x-2\sqrt{y},\\ v=x+2\sqrt{y}\end{cases}$ 变换方程

$$\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}=\frac{1}{2}\frac{\partial z}{\partial y},\quad y>0。$$

解　经计算，有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}，\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{\sqrt{y}}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)，$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}，\quad \frac{\partial^2 z}{\partial y^2}=-\frac{1}{2\sqrt{y^3}}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+\frac{1}{y}\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}\right)。$$

代入原方程，得到

$$\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}-\frac{1}{2}\frac{\partial z}{\partial y}=4\frac{\partial^2 z}{\partial u\partial v}=0，$$

所以原方程变换为

$$\frac{\partial^2 z}{\partial u\partial v}=0。$$

12．导出新的因变量关于新的自变量的偏导数所满足的方程：

（1）用$\begin{cases}u=x^2+y^2,\\ v=\dfrac{1}{x}+\dfrac{1}{y}\end{cases}$ 及$w=\ln z-(x+y)$变换方程

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)z；$$

（2）用$\begin{cases}u=x,\\ v=x+y\end{cases}$及$w=x+y+z$变换方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\left(1+\frac{y}{x}\right)\frac{\partial^2 z}{\partial y^2}=0；$$

（3）用$\begin{cases}u=x+y,\\ v=\dfrac{y}{x}\end{cases}$及$w=\dfrac{z}{x}$变换方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0。$$

解　（1）由$w=\ln z-(x+y)$得到

$$\frac{\partial z}{\partial x}=z\left(\frac{\partial w}{\partial x}+1\right)=z\left(2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}+1\right),$$

$$\frac{\partial z}{\partial y}=z\left(\frac{\partial w}{\partial y}+1\right)=z\left(2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}+1\right)。$$

代入

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)z,$$

得到

$$z\left(2xy\frac{\partial w}{\partial u}-\frac{y}{x^2}\frac{\partial w}{\partial v}+y\right)-z\left(2xy\frac{\partial w}{\partial u}-\frac{x}{y^2}\frac{\partial w}{\partial v}+x\right)-z(y-x)=0,$$

化简后得到

$$\frac{\partial w}{\partial v}=0。$$

(2) 由 $w=x+y+z$ 得到

$$\frac{\partial z}{\partial x}=\frac{\partial w}{\partial x}-1=\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}-1,\quad \frac{\partial z}{\partial y}=\frac{\partial w}{\partial y}-1=\frac{\partial w}{\partial v}-1,$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 w}{\partial u^2}+2\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2},\quad \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2},\quad \frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 w}{\partial v^2}。$$

代入

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\left(1+\frac{y}{x}\right)\frac{\partial^2 z}{\partial y^2}=0,$$

得到

$$\frac{\partial^2 w}{\partial u^2}+(\frac{v}{u}-1)\frac{\partial^2 w}{\partial v^2}=0。$$

(3) 由 $w=\frac{z}{x}$ 得到

$$\frac{\partial z}{\partial x}=w+x\frac{\partial w}{\partial x}=w+\left(x\frac{\partial w}{\partial u}-\frac{y}{x}\frac{\partial w}{\partial v}\right),\quad \frac{\partial z}{\partial y}=x\frac{\partial w}{\partial y}=x\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial w}{\partial u}-\frac{y}{x^2}\frac{\partial w}{\partial v}+\left(\frac{\partial w}{\partial u}+\frac{y}{x^2}\frac{\partial w}{\partial v}+x\frac{\partial^2 w}{\partial u^2}-\frac{2y}{x}\frac{\partial^2 w}{\partial u\partial v}+\frac{y^2}{x^3}\frac{\partial^2 w}{\partial v^2}\right)$$

$$=2\frac{\partial w}{\partial u}+x\frac{\partial^2 w}{\partial u^2}-\frac{2y}{x}\frac{\partial^2 w}{\partial u\partial v}+\frac{y^2}{x^3}\frac{\partial^2 w}{\partial v^2}，$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial w}{\partial u}+x\frac{\partial^2 w}{\partial^2 u}+(1-\frac{y}{x})\frac{\partial^2 w}{\partial u\partial v}-\frac{y}{x^2}\frac{\partial^2 w}{\partial v^2}，$$

$$\frac{\partial^2 z}{\partial y^2}=x\frac{\partial^2 w}{\partial u^2}+2\frac{\partial^2 w}{\partial u\partial v}+\frac{1}{x}\frac{\partial^2 w}{\partial v^2}。$$

代入

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0，$$

得到

$$\frac{\partial^2 w}{\partial v^2}=0。$$

13．设 $y=f(x,t)$ ，而 t 是由方程 $F(x,y,t)=0$ 所确定的 x,y 的隐函数，其中 f 和 F 都具有连续偏导数。证明

$$\frac{dy}{dx}=\frac{\frac{\partial f}{\partial x}\frac{\partial F}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial F}{\partial x}}{\frac{\partial f}{\partial t}\frac{\partial F}{\partial y}+\frac{\partial F}{\partial t}}。$$

证　设由方程 $F(x,y,t)=0$ 所确定的隐函数为 $t=h(x,y)$ ，于是就由方程 $y=f(x,t)=f\left(x,h(x,y)\right)$ 确定了隐函数 $y=y(x)$ ，并由此可知 t 也是 x 的一元函数，即 $t=h\left(x,y(x)\right)=t(x)$ 。

首先在等式 $F(x,y,t)=F\left(x,y(x),t(x)\right)=0$ 两边对 x 求导，得到

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial t}\frac{dt}{dx}=0，$$

解出

$$\frac{dt}{dx}=-\frac{\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}}{\frac{\partial F}{\partial t}}，$$

然后再在等式 $y=f\left(x,t(x)\right)$ 两边对 x 求导，得到

$$\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}\frac{dt}{dx}=\frac{\partial f}{\partial x}-\frac{\partial f}{\partial t}\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}\right)\left(\frac{\partial F}{\partial t}\right)^{-1},$$

从而解出

$$\frac{dy}{dx}=\frac{\frac{\partial f}{\partial x}\frac{\partial F}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial F}{\partial x}}{\frac{\partial f}{\partial t}\frac{\partial F}{\partial y}+\frac{\partial F}{\partial t}}。$$

14. 设二元函数 $f(x,y):\mathbf{R}^2\to\mathbf{R}$ 具有连续偏导数，证明：存在一对一的连续的向量值函数 $\mathbf{G}(t):\mathbf{R}\to\mathbf{R}^2$，使得

$$f\circ\mathbf{G}\equiv\text{常数}。$$

证 若函数 $f(x,y)$ 恒等于常数，则任意的一对一的连续的向量值函数 $\mathbf{G}(t):\mathbf{R}\to\mathbf{R}^2$ （例如 $\mathbf{G}(t)=(t,t)$ ）都满足要求。

现假设函数 $f(x,y)$ 不恒等于常数，则存在 (x_0,y_0)，使得 $f_x(x_0,y_0)$ 和 $f_y(x_0,y_0)$ 不全为 0，不妨设 $f_y(x_0,y_0)\neq 0$。记 $F(x,y)=f(x,y)-f(x_0,y_0)$，它满足定理 12.4.1 的所有条件，所以在 x_0 的邻域 (a,b) 存在严格单调的连续函数 $y=g(x)$ 满足 $F(x,g(x))\equiv 0$，即 $f(x,g(x))\equiv$ 常数。

设 $t=\tan\pi\left(\frac{x-a}{b-a}-\frac{1}{2}\right)$ 的逆函数为 $x=x(t):(-\infty,+\infty)\to(a,b)$，则

$$\mathbf{G}(t)=\big(x(t),g(x(t))\big)$$

是 $\mathbf{R}\to\mathbf{R}^2$ 的一对一的连续的向量值函数，满足题目要求。